

On Coloring Random Subgraphs of a Fixed Graph

Igor Shinkar

igors@berkeley.edu

UC Berkeley

December 14, 2016

Abstract

Given a graph G we study the chromatic number of a random graph $G_{1/2}$ obtained from G by removing each edge of G independently with probability $1/2$. Studying $\chi(G_{1/2})$ has been suggested by Bukh [Buk], who asked whether $\mathbb{E}[\chi(G_{1/2})] \geq \Omega(\chi(G)/\log(\chi(G)))$ holds for all graphs G . In this paper we prove several results related this problem. Denoting the chromatic number of G by $k = \chi(G)$ we prove the following results.

1. For all $d \leq k^{1/3}$ it holds that $\Pr[\chi(G_{1/2}) \leq d] < \exp\left(-\Omega\left(\frac{k(k-d^3)}{d^3}\right)\right)$. In particular, $\Pr[G_{1/2} \text{ is bipartite}] < \exp(-\Omega(k^2))$. When G is the k -clique, this bound is tight up to a constant in $\Omega(\cdot)$.
2. For all $\varepsilon > 0$ it holds that $\Pr[\chi(G_{1/2}) \leq (1 - \varepsilon)\sqrt{k}] < \exp\left(-\Omega\left(\varepsilon^2\sqrt{k}\right)\right)$.

We also prove that for graphs G with $\chi(G) = k$ and $\alpha(G) \leq O(n/k)$, it holds that $\mathbb{E}[\chi(G_{1/2})] \geq \Omega(k/\log(k))$.

1 Introduction

For a given graph G let G_p be a random subgraph of G obtained from it by removing each edge of G independently with probability $1 - p$. In this paper we study the chromatic number of $G_{1/2}$ for an arbitrary graph G whose chromatic number is equal to some parameter k . Clearly, since $G_{1/2}$ is a subgraph of G , it holds that $\chi(G_{1/2}) \leq \chi(G)$. If G is the k -clique, then this is the well studied Erdős-Rényi random graph model [ER60], where it is known that $\chi(G_{1/2}) = \Theta(\frac{k}{\log(k)})$ with high probability (see, e.g. [Bol01]). By monotonicity we also see that if $\chi(G) = k$ and G contains a k -clique, then with high probability $\chi(G_{1/2}) \geq \Omega(\frac{k}{\log(k)})$. It is not difficult to come up with an example of a graph G for which $\chi(G_{1/2}) = \chi(G)$ with high probability. (For instance, let G be the complete k -partite graph with $\text{poly}(k)$ vertices in each part.) Given the foregoing examples, it is natural to ask whether for every graph G

if $\chi(G)$ is large, then $\chi(G_{1/2})$ is also large with high probability. Studying $\chi(G_{1/2})$ has been suggested by Bukh [Buk], who asked the following question.

Is there a constant $c > 0$ such that $\mathbb{E}[\chi(G_{1/2})] > c \cdot \frac{\chi(G)}{\log \chi(G)}$ for all G ?

Recently there has been some work generalizing the classical result on random graphs, asking about properties of random subgraphs of a fixed graph, or of graphs satisfying certain properties [BCvdH⁺05a, BCvdH⁺05b, FK13, KLS15]. For example, there have been several results studying the emergence of a giant component in G_p when G is an expander graph [KS13, ABS04, FKM04]. In this work we study a problem of a similar flavor, namely, trying to relate the chromatic random of G_p to the chromatic number of a random graph in the Erdős-Rényi random model.

In a slightly different context, this problem is also motivated by a recent work of Bennett et al. [BRS16], who asked about the computational complexity of \mathcal{NP} -complete problems, whose inputs come from a certain semi-random model. In particular, they showed that many natural \mathcal{NP} -complete problems, such as finding the chromatic number of a graph, or deciding whether a graph contains a Hamiltonian path, remain NP -hard even in the seemingly relaxed situation, where the inputs to the problem come from random subgraphs of worst case instances. In particular, they proved that if $\chi(G) = k$, then $\Pr[\chi(G_{1/2}) < d] < \text{poly}(\frac{1}{k-d^3})$ for all $d < k^{1/3}$.

1.1 Our results

The main result in this paper gives an upper bound on the probability that $\chi(G_{1/2})$ is very small compared to $\chi(G)$.

Theorem 1.1. *Let $G = (V, E)$ be a graph with $\chi(G) = k$.*

1. *For all $d \leq k^{1/3}$ it holds that $\Pr[\chi(G_{1/2}) \leq d] < \exp\left(-\Omega\left(\frac{k(k-d^3)}{d^3}\right)\right)$.*
2. *For all $\varepsilon > 0$ it holds that $\Pr[\chi(G_{1/2}) \leq (1 - \varepsilon)\sqrt{k}] < \exp\left(-\Omega\left(\varepsilon^2\sqrt{k}\right)\right)$.*

Remark. For $d = 2$ in Item 1 we get that $\Pr[G_{1/2} \text{ is bipartite}] < \exp(-\Omega(k^2))$. When G is the k -clique this bound is best possible up to a constant in the $\Omega(\cdot)$.

Next, we study $\chi(G_{1/2})$ for a special (rather large) class of graphs. Note that if G is an n -vertex graph with $\chi(G) = k$, then G contains an independent set of size n/k . However, in many cases the maximal independent set of G is within a multiplicative constant factor of n/k , i.e., $\alpha(G) \leq C \cdot \frac{n}{k}$ for some $C > 1$ that is not too large. For example, the random graph models $G(n, p)$ and $G(n, d)$ satisfy this property with high probability for all $p > \frac{1}{n}$ and $d \geq 2$ (see, e.g., [Bol01]).

Theorem 1.2. *Let $G = (V, E)$ be a graph with $\alpha(G) \leq C \cdot \frac{n}{k}$ for some $C > 1$. Then*

$$\Pr\left[\alpha(G_{1/2}) \geq \frac{4C \log(k)}{k} n\right] > 1 - 2^{-\frac{4C \log(k)}{k} n}.$$

In particular,

$$\mathbb{E}[\alpha(G_{1/2})] \leq \frac{k}{8C \log(k)}.$$

The following corollary follows immediately from Theorem 1.2.

Corollary 1.3. *Let $G = (V, E)$ be a graph with $\chi(G) = k$, and suppose that G contains a subgraph $G' = (V', E')$ with $V' \subseteq V$ such that $\alpha(G') \leq C \frac{|V'|}{k}$. Then,*

$$\mathbb{E}[\chi(G_{1/2})] \geq \frac{k}{8C \log(k)}.$$

Next, we discuss a related graph parameter, called the *Hadwiger number* of a graph. Hadwiger number of a graph G , denoted by $h(G)$, is the maximal $t \in \mathbb{N}$ such that G contains K_t as a minor. Hadwiger's conjecture states that $h(G) \geq \chi(G)$ for all graphs G . While the conjecture is open for general graphs, the inequality $h(G) \geq \chi(G)$ is known to hold for a random graph $G(n, 1/2)$ with high probability. Mader [Mad68] proved an approximate version of the conjecture, namely that $h(G) \geq \Omega\left(\frac{\chi(G)}{\log(k)}\right)$ for all graphs G .

Kostochka [Kos84] improved Mader's result, and showed that $h(G) \geq \Omega\left(\frac{\chi(G)}{\sqrt{\log(k)}}\right)$ for all graphs G . Motivated by this line of research Adrian Vetta [Vet16] asked the following question. What is $\min\{\mathbb{E}[h(G_{1/2})] : G \text{ such that } \chi(G) = k\}$? We note that a tight answer to this question follows almost immediately from [Kos84].

Theorem 1.4. *Let $G = (V, E)$ be an n -vertex graph with $\chi(G) = k$. Then,*

$$\Pr\left[h(G) \geq \Omega\left(\frac{k}{\sqrt{\log(k)}}\right)\right] \geq 1 - \exp(-\Omega(k^2)).$$

In particular, $\mathbb{E}[h(G_{1/2})] \geq \Omega\left(\frac{k}{\sqrt{\log(k)}}\right)$.

Remark. By the result of [BCE80] when G is the k -clique we have $\mathbb{E}[h(G_{1/2})] \leq O\left(\frac{k}{\sqrt{\log(k)}}\right)$, and so the bound in Theorem 1.4 result is tight up to a multiplicative constant.

2 Preliminaries

Let $G = (V, E)$ be an undirected graph. An independent set in G is a subset of the vertices that spans no edges. The *independence number* of G , denoted by $\alpha(G)$, is the largest size of an independent set in G . A vertex coloring of G is an assignment of colors to V such that no two adjacent vertices have the same color. The chromatic number of G , denoted by $\chi(G)$, is the smallest number of colors required to color G . Note that in any vertex coloring of G

each color class forms an independent set, and hence $\alpha(G) \geq n/\chi(G)$. Hadwiger number of a graph G , denoted by $h(G)$, is the maximal $t \in \mathbb{N}$ such that G contains K_t as a minor.

For a subset of the vertices $A \subseteq V$ let $E(A)$ be the set of edges spanned by A , i.e., $E(A) = \{(u, v) \in E : u, v \in A\}$. For two disjoint subsets of the vertices $A, B \subseteq V$ define $\text{cut}(A, B) = \{(u, v) \in E : u \in A, v \in B\}$ to be the set of edges with one endpoint in A and one endpoint in B .

We will need the following easy claim saying that the number of edges in a graph is at least quadratic in its chromatic number.

Claim 2.1. *Let $G = (V, E)$ be a graph with chromatic number $\chi(G) = k$. Then $|E| \geq \binom{k}{2}$.*

Proof. Let $V = C_1 \cup \dots \cup C_k$ be a partition of the vertices of G into k color classes. Note that there must be at least one edge between every two color classes, as otherwise, if there are no edges between C_i and C_j , then $C_i \cup C_j$ is an independent set, and we can color them with the same color, which implies that $\chi(G) \leq k - 1$. Therefore $|E| \geq \binom{k}{2}$. \square

We will also need a result from extremal set theory about r -wise t -intersecting families. In order to explain the result we will need some notation. Let X be a finite set, and let $P(X) = \{F : F \subseteq X\}$ be the collection of all subsets of X . A family of sets $\mathcal{F} \subseteq P(X)$ is said to be r -wise t -intersecting if for every $F_1, \dots, F_r \in \mathcal{F}$ it holds that $|F_1 \cap F_2 \cap \dots \cap F_r| \geq t$. In the proof of Theorem 1.1 we will use the following theorem due to Frankl [Fra87].

Theorem 2.2 (Claim 9.2 [Fra87]). *Let $\mathcal{F} \subseteq P(X)$ be a 3-wise t -intersecting family. Then $|\mathcal{F}| < \left(\frac{\sqrt{5}-1}{2}\right)^t \cdot 2^n$.*

3 Probability that $\chi(G_{1/2})$ is Small

In this section we prove Theorem 1.1. We start with the proof of Item 1 for the case of $d = 2$, as we think it is a bit cleaner than the proof for general d .

Proof of Item 1 for $d = 2$. Let us assume that $k > 8$, as otherwise the claim holds trivially. Let $(A, V \setminus A)$ be a partition of the vertices of G into 2 parts. Define $\text{uncut}(A) = E(A) \cup E(V \setminus A)$ to be all the edges that do not belong to $\text{cut}(A, V \setminus A)$. Let $S \subseteq E$ be a random subset of the edges, where each $e \in E$ is added to S independently with probability $1/2$, and let $H = (V, E \setminus S)$ so that H is distributed according to $G_{1/2}$. Note that H is bipartite if and only if there exists some $A \subseteq V$ such that $\text{uncut}(A) \subseteq S$.

$$\Pr[H \text{ is bipartite}] = \Pr[\exists A : \text{uncut}(A) \subseteq S]. \quad (1)$$

Let U be the monotone closure of $\{\text{uncut}(A) : A \subseteq V\}$ defined as

$$U = \{S \subseteq E : \exists A \subseteq V \text{ such that } \text{uncut}(A) \subseteq S\}.$$

Using this notation we have $\Pr[H \text{ is bipartite}] = \Pr[S \in U] = \frac{|U|}{2^{|E|}}$, and hence, it is enough to show an upper bound on $|U|$. We make the following observation:

Observation 3.1. U is a 3-wise t -intersecting family for some $t \geq \Omega(k^2)$.

Proof. Note that by monotonicity of U it is enough to show that for any $A_1, A_2, A_3 \subseteq V$ it holds that $|\text{uncut}(A_1) \cap \text{uncut}(A_2) \cap \text{uncut}(A_3)| \geq \Omega(k^2)$. Indeed, the three subsets A_1, A_2, A_3 partition V into 8 parts V_1, \dots, V_8 according to whether a vertex belongs to A_i or to its complement. Denote by m_i the number of edges spanned by V_i , and denote by k_i the chromatic number of $G[V_i]$. Then, $\sum_{i=1}^8 k_i \geq k$, and hence for some i^* we must have $k_{i^*} \geq k/8$. On the other hand, by Claim 2.1 we have $m_{i^*} \geq \binom{k_{i^*}}{2} \geq \binom{k/8}{2} \geq \Omega(k^2)$, which immediately implies that $|\text{uncut}(A_1) \cap \text{uncut}(A_2) \cap \text{uncut}(A_3)| \geq m_{i^*} \geq \Omega(k^2)$, as required. \square

Therefore, by applying Theorem 2.2 we conclude that $|U| \leq (\frac{\sqrt{5}-1}{2})^t \cdot 2^{|E|}$, and hence

$$\Pr[H \text{ is bipartite}] = \Pr[S \in U] = \frac{|U|}{2^{|E|}} \leq \exp(-\Omega(k^2)),$$

as required. \square

Next, we generalize the proof above and prove Item 1 of Theorem 1.1.

Proof of Item 1. Let $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_d)$ be a partition of the vertices of G into d classes, i.e., the \mathcal{A}_i 's are pairwise disjoint and $V = \mathcal{A}_1 \cup \dots \cup \mathcal{A}_d$. For such a partition \mathcal{A} define $\text{uncut}(\mathcal{A}) = E(\mathcal{A}_1) \cup \dots \cup E(\mathcal{A}_d) \subseteq E$ to be the set of the edges of G with both endpoints in some \mathcal{A}_i . Denote by $\mathcal{P}_d = \{\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_d)\}$ the collection of all such partitions of the vertices into d classes.

Let $H \sim G_{1/2}$, and let $S \subseteq E$ be a random subset of the edges, where each $e \in E$ is added to S independently with probability $1/2$. Note that by considering $H = (V, E \setminus S)$ we have

$$\Pr[\chi(H) \leq d] = \Pr[\exists \mathcal{A} \in \mathcal{P}_d : \text{uncut}(\mathcal{A}) \subseteq S]. \quad (2)$$

Let U be the monotone closure of $\{\text{uncut}(\mathcal{A}) : \mathcal{A} \in \mathcal{P}_d\}$ defined as

$$U = \{\text{uncut}(\mathcal{A}) : \mathcal{A} \in \mathcal{P}_d\}^\uparrow = \{S \subseteq E : \exists \mathcal{A} \in \mathcal{P}_d \text{ such that } \text{uncut}(\mathcal{A}) \subseteq S\}.$$

Then $\Pr[\chi(H) \leq d] = \Pr[\exists \mathcal{A} : S \supseteq \text{uncut}(\mathcal{A})] = \frac{|U|}{2^{|E|}}$, and hence, it is enough to show an upper bound on $|U|$. The key step is the following claim.

Claim 3.2. U is a 3-wise t -intersecting family for $t = \lceil \frac{k(k-d^3)}{2d^3} \rceil$.

We postpone the proof of the claim, and show how it implies the theorem. By applying Theorem 2.2 we conclude that $|U| \leq (\frac{\sqrt{5}-1}{2})^t \cdot 2^{|E|}$, and hence

$$\Pr[\chi(H) \leq d] = \Pr[\exists \mathcal{A} : S \supseteq \text{uncut}(\mathcal{A})] = \frac{|U|}{2^{|E|}} \leq \left(\frac{\sqrt{5}-1}{2}\right)^t \leq 0.79^{\frac{k(k-d^3)}{d^3}},$$

as required. \square

We now return to the proof of Claim 3.2.

Proof. Note that since U is a monotone closure of $\{\text{uncut}(\mathcal{A}) : \mathcal{A} \in \mathcal{P}_d\}$ it is enough to show that $\{\text{uncut}(\mathcal{A}) : \mathcal{A} \in \mathcal{P}_d\}$ is a 3-wise t -intersecting family. Indeed, let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{P}_d$ be three partitions, and let $t = |\text{uncut}(\mathcal{A}) \cap \text{uncut}(\mathcal{B}) \cap \text{uncut}(\mathcal{C})|$ be the size of the intersection. Let V_1, \dots, V_{d^3} be the partition V according to the partitions \mathcal{A}, \mathcal{B} , and \mathcal{C} , and for each $i \in \{1, \dots, d^3\}$ let $t_i = |E[V_i]|$ and $k_i = \chi(G[V_i])$. Note that (1) $t = \sum_{i=1}^{d^3} t_i$ and (2) $k \leq \sum_{i=1}^{d^3} k_i$ since we color each V_i with k_i new colors then we will obtain a legal coloring of G . Therefore,

$$t \stackrel{(1)}{=} \sum_{i=1}^{d^3} t_i \stackrel{(*)}{\geq} \sum_{i=1}^{d^3} \binom{k_i}{2} \stackrel{(**)}{\geq} d^3 \left(\frac{1}{d^3} \sum_i k_i \right) \stackrel{(2)}{\geq} d^3 \left(\frac{k}{d^3} \right) = \frac{k(k - d^3)}{2d^3},$$

where $(*)$ is by Claim 2.1, and $(**)$ is by Jensen's inequality, using the fact that $x \mapsto \frac{x(x-1)}{2}$ is a convex function. This completes the proof of Claim 3.2. \square

We now turn to the proof of Item 2 of Theorem 1.1. We start with the following easy observation, also made by Bukh [Buk].

Observation 3.3. *If $\chi(G) = k$, then $\mathbb{E}[\chi(G)_{1/2}] \geq \sqrt{k}$.*

Proof. Let $G = (V, E)$. Let $H = (V, E_H)$ be a subgraph of G and let $\overline{H} = (V, E \setminus E_H)$ be the complement of H in G . Note that $\chi(H) \cdot \chi(\overline{H}) \geq k$. Indeed, if $c_H : V \rightarrow [\chi(H)]$ is a coloring of H and $c_{\overline{H}} : V \rightarrow [\chi(\overline{H})]$ is a coloring of \overline{H} , then we can construct a coloring c_G of G with at most $\chi(H) \cdot \chi(\overline{H})$ colors by letting $c_G(v) = (c_H(v), c_{\overline{H}}(v))$. To see that c_G is indeed a legal coloring note that every edge (u, v) in G belongs to either H or \overline{H} , and hence $c_G(u)$ differs from $c_G(v)$ is at least one of the coordinates. This implies that $\frac{\chi(H) + \chi(\overline{H})}{2} \geq \sqrt{\chi(H) \cdot \chi(\overline{H})} \geq \sqrt{k}$ for all subgraphs $H \subseteq G$. Taking expectation of the inequality above, and recalling that each of H and \overline{H} are distributed like $G_{1/2}$ we get that

$$\mathbb{E}[\chi(G)_{1/2}] = \frac{\mathbb{E}[\chi(H)] + \mathbb{E}[\chi(\overline{H})]}{2} \geq \sqrt{k}.$$

\square

Next, we claim that $\chi(G_{1/2})$ is concentrated around its expectation.

Lemma 3.4. *Let $G = (V, E)$ be a graph with $\chi(G) = k$. Then for all $\varepsilon > 0$ we have $\Pr[\chi(G_{1/2}) < (1 - \varepsilon) \cdot \mathbb{E}[\chi(G_{1/2})]] < e^{-\Omega(\varepsilon^2 \cdot \mathbb{E}[\chi(G_{1/2})])}$.*

Proof. Let $V = C_1 \cup \dots \cup C_k$ be a partition of the vertices of G into k color classes, and let $H = G_{1/2}$. Let X_1, X_2, \dots, X_k be a sequence of random variables defined by $X_i = \chi(H[C_i])$, i.e. X_i is the chromatic number of the subgraph of H induced by C_i . In particular $\mathbb{E}[X_k] = \mathbb{E}[\chi(G)_{1/2}]$. Note that since each C_i is an independent set in G , it is also an independent set in H , and hence $X_{i+1} - X_i \in \{0, 1\}$.

We now use the following large deviation result due to Alon et al. [AGGL10], which, in turn relies on the Bernstein-Kolmogorov type inequality of Freedman [Fre75].

Proposition 3.5. *Let X_1, \dots, X_k sequence of random variables adapted to some filter (\mathcal{F}_i) such that $|X_{i+1} - X_i| \leq 1$ for all $i = 1, \dots, k-1$. Then,*

$$\Pr \left[\left| \frac{X_k}{\mathbb{E}[X_k]} - 1 \right| > \varepsilon \right] < O(e^{-\Omega(\varepsilon^2 \mathbb{E}[X_k])})$$

By Proposition 3.5 we have

$$\Pr[\chi(G_{1/2}) < (1 - \varepsilon) \cdot \mathbb{E}[\chi(G_{1/2})]] = \Pr \left[\frac{X_k}{\mathbb{E}[X_k]} < 1 - \varepsilon \right] < O(e^{-\Omega(\varepsilon^2 \mathbb{E}[X_k])}).$$

□

Proof of Item 2. The proof follows immediately from Observation 3.3 and Lemma 3.4. Indeed, by Observation 3.3 we have $\mathbb{E}[\chi(G_{1/2})] \geq \sqrt{k}$, and by applying Lemma 3.4 we get that for all $\varepsilon > 0$ it holds that

$$\Pr[\chi(G_{1/2}) < (1 - \varepsilon) \cdot \sqrt{k}] < e^{-\Omega(\varepsilon^2 \cdot \sqrt{k})},$$

as required. □

4 Proofs of Theorem 1.2 and Corollary 1.3

Theorem 1.2 has been essentially proven in [BRS16] Theorem 3. We reproduce the proof here mostly for completeness.

Proof of Theorem 1.2. Let $G = (V, E)$ be a graph with $\chi(G) = k$ and $\alpha(G) \leq C \cdot \frac{n}{k}$ for some $C > 1$. We will need the claim, also known as Turán's theorem [Tur41].

Claim 4.1. *Let G be a graph with maximal independent set of size r . Then for every $t > 1$ every set of vertices of size tr spans at least $\frac{t(t-1)}{2}r$ edges.¹*

In our setting we have $\alpha(G) \leq \frac{C}{k}n$, and hence, every set of $\ell = \frac{4C \log(k)}{k}n$ vertices spans at least $\frac{\ell(\ell/\alpha(G)-1)}{2} > \frac{\ell^2}{4\alpha} \geq \frac{4C \log(k)^2}{k}n$ edges. Let $S \subseteq V$ be a subset of the vertices of size $|S| = \ell = \frac{4C \log(k)}{k}n$. Then the probability that S is an independent set in $G_{1/2}$ is at most $2^{|E[S]|} \leq 2^{-\frac{4C \log(k)^2}{k}n}$. Therefore, by taking union bound over all subsets of size ℓ we get that

$$\Pr[\alpha(G_{1/2}) \geq \ell] \leq \binom{n}{\ell} \cdot 2^{-\frac{4C \log(k)^2}{k}n} \leq \left(\frac{n \cdot e}{\ell}\right)^\ell \cdot k^{-\ell} = \left(\frac{n \cdot e}{\ell k}\right)^\ell < 2^{-\ell}.$$

In particular, $\Pr[\chi(G_{1/2}) \leq \frac{k}{4C \log(k)}] \leq \Pr[\alpha(G_{1/2}) \geq \ell] < 2^{-\ell} < 1/2$. This implies that $\mathbb{E}[\chi(G_{1/2})] > \frac{k}{8C \log(k)}$, and the theorem follows. □

¹ We remark that Turán's theorem is usually stated as follows: If an n vertex graph G does not contain an $(r+1)$ -clique, then the number of edges in G is at most $\frac{r-1}{2r}n^2$ edges. The statement of Claim 4.1 can be easily derived by considering the complement graph.

Next we turn to proving Corollary 1.3.

Proof of Corollary 1.3. Let G' be the subgraph as in the assumption. By Theorem 1.2 we have

$$\mathbb{E}[\chi(G_{1/2})] > \mathbb{E}[\chi(G'_{1/2})] > \frac{k}{8C \log(k)},$$

as required. \square

5 Proof of Theorem 1.4

The theorem follows almost immediately from the following result of Kostochka [Kos84].

Theorem 5.1 ([Kos84] Theorem 1). *Let $G = (V, E)$ be a graph such that $|E| \geq k \cdot |V|$. Then $h(G) \geq \Omega\left(\frac{k}{\sqrt{\log(k)}}\right)$.*

Proof of Theorem 1.4. Suppose that k is sufficiently large (e.g., $k \geq 10$), as otherwise the theorem holds trivially. Let G be an n vertex graph with $\chi(G) = k$. Let $G' = (V', E')$ be a k -critical subgraph of G , i.e., G' is a subgraph of G such that $\chi(G') = k$ but removing any edge from G' reduces its chromatic number. Then, every vertex of G' has degree at least $k - 1$, and hence, $|V'| \geq k$ and $|E'| \geq \frac{k-1}{2}|V'| \geq \frac{k}{4}|V'|$. Let $H = (V', E_H) \sim G'_{1/2}$. Then, by Chernoff bound we have $\Pr[|E_H| < k \cdot |V'|/8] < \exp(-\Omega(k \cdot |V'|)) < \exp(-\Omega(k^2))$. Applying Theorem 5.1 to H we get that $h(H) \geq \Omega\left(\frac{k}{\sqrt{\log(k)}}\right)$ with probability $1 - \exp(-\Omega(k^2))$. This completes the proof of Theorem 1.4. \square

6 Open Problems

The most obvious open problem in this context is the original question of Bukh.

Question 6.1. *Is there a constant $c > 0$ such that $\mathbb{E}[\chi(G_{1/2})] > c \cdot \frac{\chi(G)}{\log \chi(G)}$ for all G ?*

Note that by the martingale argument from Lemma 3.4 a positive answer would follow from the (seemingly weaker) bound $\Pr[\chi(G_{1/2}) > \Omega(\frac{\chi(G)}{\log \chi(G)})] > \exp(-\Omega(\frac{\chi(G)}{\log \chi(G)}))$. Other than Bukh's original question, this paper raises several additional problems which we mention below.

Question 6.2. *Is it true that every graph G contains an induced subgraph $G' \subseteq G$ such that $\chi(G') \geq c \cdot \chi(G)$, and $\alpha(G') \leq C \frac{|V(G')|}{\chi(G')}$ for some absolute constants $C, c > 0$?*

A positive answer to this question would immediately give a positive answer to Bukh's question using Theorem 1.2. We stress that Question 6.2 does not require any conditions on the number of vertices on G' , except for the obvious $|V(G')| \geq \chi(G') \cdot \alpha(G')/C$.

Recall that $G(k, 1/2)$ is the random Erdős-Rényi graph model [ER60] obtained from the k -clique where each of the clique is kept with probability $1/2$. A routine calculation shows that in this case we have $\Pr[\chi(G(k, 1/2)) \leq d] < e^{-\Omega(\frac{k(k-d \log(d))}{d})}$ for all $d < ck/\log(k)$.² It seems natural to ask whether it is possible to compare $\chi(G(k, 1/2))$ with $\chi(G_{1/2})$ for an arbitrary G with $\chi(G) = k$. In particular, is it true that for all graphs G with $\chi(G) = k$ and for all $d < k$ it holds that $\Pr[\chi(G_{1/2}) \leq d] < \Pr[\chi(G(k, 1/2)) \leq d]$? However, it is not difficult to see that for $k = 3$ this is false by taking $G = C_n$ to be the odd length cycle of length n . Indeed, in this case $\Pr[\chi(G_{1/2}) \leq 2] = 1 - 2^n$, while $\Pr[\chi(G(k, 1/2)) \leq 2] = 7/8$. This example can be easily extended to any k and $d = 2k/3$; we omit the details. Still, it is natural to ask whether a slightly relaxed comparison is true. We ask a slightly more general question about G_p for an arbitrary $p \in (0, 1)$.

Question 6.3. *Let $p \in (0, 1)$. Is there a constant $C > 1$ and a polynomial $\text{poly} : \mathbb{R} \rightarrow \mathbb{R}$ such that $\Pr[\chi(G_p) \leq d] < \text{poly}(\Pr[\chi(G(k, p)) \leq C \cdot d])$ holds for every graph G , and for all $d \leq k$, where $k = \chi(G)$ and $G(k, p)$ is the random Erdős-Rényi graph model?*

Note that Theorem 1.1 gives a positive answer to Question 6.3 for $p = 1/2$ and $d = O(1)$.

We conclude with two more problems that we find interesting.

Question 6.4. *Let $p \in (0, 1/2)$. Is there a constant $c > 0$ such that $\mathbb{E}[\chi(G_p)] \geq c \cdot \chi(G)^p$ for all G ?*

Question 6.5. *Let $p \in (0, 1)$. Is there a constant $c > 0$ such that $\mathbb{E}[\chi(G_{p/2})] \geq c \cdot \mathbb{E}[\chi(G_p)]$ for all G ?*

7 Acknowledgements

I am grateful to Huck Bennett for many helpful discussions related to this work. Huck had several crucial observations related to this work, but persistently refused to co-author the paper. I am thankful to Daniel Reichman for valuable pointers to the literature and for many discussions related to this paper. I am also thankful to Boris Bukh for his helpful comments.

References

- [ABS04] Noga Alon, Itai Benjamini, and Alan Stacey. Percolation on finite graphs and isoperimetric inequalities. *Ann. Probab.*, 32(3):1727–1745, 07 2004.
- [AGGL10] Noga Alon, Ori Gurel-Gurevich, and Eyal Lubetzky. Choice-memory tradeoff in allocations. *The Annals of Applied Probability*, 20(4):1470–1511, 2010.

²For $d < \frac{k}{2 \log(k)}$, and for any partition of the k vertices into d classes, the k -clique has at least $k^2/4d$ edges that belong to one of the color classes, and hence, by taking union bound over all possible d -colorings we have $\Pr[\chi(G(k, 1/2)) \leq d] \leq d^k \cdot 2^{-k^2/4d} = 2^{-\Omega(\frac{k(k-4d \log(d))}{4d})}$.

- [BCE80] B. Bollobás, P.A. Catlin, and P. Erdős. Hadwiger’s conjecture is true for almost every graph. *European Journal of Combinatorics*, 1(3):195 – 199, 1980.
- [BCvdH⁺05a] Christian Borgs, Jennifer T. Chayes, Remco van der Hofstad, Gordon Slade, and Joel Spencer. Random subgraphs of finite graphs: I. the scaling window under the triangle condition. *Random Structures & Algorithms*, 27(2):137–184, 2005.
- [BCvdH⁺05b] Christian Borgs, Jennifer T. Chayes, Remco van der Hofstad, Gordon Slade, and Joel Spencer. Random subgraphs of finite graphs. ii. the lace expansion and the triangle condition. *The Annals of Probability*, 33(5):1886–1944, 2005.
- [Bol01] Béla Bollobás. *Random graphs*. Cambridge University Press, 2001.
- [BRS16] Huck Bennett, Daniel Reichman, and Igor Shinkar. On Percolation and NP-Hardness. In *43rd International Colloquium on Automata, Languages, and Programming (ICALP 2016)*, 2016.
- [Buk] Boris Bukh. Interesting problems that I cannot solve. Problem 2. <http://www.borisbukh.org/problems.html>.
- [ER60] P. Erdős and A. Rényi. On the evolution of random graphs. *Publications of the Mathematical Institute of the Hungarian Academy of Sciences*, 5:17–61, 1960.
- [FK13] Alan Frieze and Michael Krivelevich. On the non-planarity of a random subgraph. *Combinatorics, Probability and Computing*, 22(5):722–732, 009 2013.
- [FKM04] Alan Frieze, Michael Krivelevich, and Ryan Martin. The emergence of a giant component in random subgraphs of pseudo-random graphs. *Random Structures & Algorithms*, 24(1):42–50, 2004.
- [Fra87] Peter Frankl. The shifting technique in extremal set theory. *Surveys in Combinatorics, London Math. Soc. Lecture Note Ser.*, 123:81–110, 1987.
- [Fre75] David A. Freedman. On tail probabilities for martingales. *Ann. Probab.*, 3(1):100–118, 1975.
- [KLS15] Michael Krivelevich, Choongbum Lee, and Benny Sudakov. Long paths and cycles in random subgraphs of graphs with large minimum degree. *Random Structures & Algorithms*, 46(2):320–345, 2015.
- [Kos84] A. V. Kostochka. A lower bound for the hadwiger number of graphs by their average degree. *Combinatorica*, pages 307–316, 1984.

- [KS13] Michael Krivelevich and Benny Sudakov. The phase transition in random graphs: A simple proof. *Random Structures & Algorithms*, 43(2):131–138, 2013.
- [Mad68] W. Mader. Homomorphiesätze für graphen. *Mathematische Annalen*, 178:154–168, 1968.
- [Tur41] Paul Turán. On an extremal problem in graph theory. *Matematikai s Fizikai Lapok* (in Hungarian) , 48:436–452, 1941.
- [Vet16] Adrian Vetta. Personal communication, 2016.